JOURNAL OF APPROXIMATION THEORY **54**, 338–353 (1988)

The Lebesgue Constant for Lagrange Interpolation in the Simplex

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Communicated by E. W. Cheney

Received August 15, 1986

1. INTRODUCTION

1.1. Let $X \subset \mathbf{R}^k$ be compact and let π_d^k denote the space of polynomials of (total) degree $\leqslant d$ in k variables. Then, $\dim \pi_d^k = \binom{k+d}{d}$ and we will denote this number by $T(d, k)$.

Lagrange interpolation is as follows: Given distinct points $A_i \in X$ for $i = 1, ..., T(d, k)$ and real numbers b_i for $i = 1, ..., T(d, k)$, then the Lagrange interpolating polynomial $L \in \pi_d^k$ is defined by

$$L(A_i) = b_i, \qquad \text{for } i = 1, ..., T(d, k). \tag{1.1.1}$$

L exists and is unique if the points $\{A_i\}$ do not satisfy a polynomial relation of degree $\leqslant d$. (We will always assume this to be the case.)

1.2. Let l_{dv} be the unique polynomial in π_d^k which satisfies Eq. (1.1.1) with

$$b_i = 0, \qquad i \neq v,$$
$$b_i = 1, \qquad i = v.$$

That is

$$l_{dv}(A_i) = \delta_{vi}, \qquad \text{for} \quad i, v = 1, ..., T(d, k). \tag{1.2.1}$$

Then

$$L(x) = \sum_{v=1}^{T(d,k)} b_v \, l_{dv}(x). \tag{1.2.2}$$

* Supported by NSERC of Canada Grant A7535.

0021-9045/88 $3.00

If f is a function defined on X the Lagrange interpolating polynomial to f, denoted $L_f(x)$, is the unique polynomial of degree $\leqslant d$ which takes on the same values as f at the points A_i, for $i=1, ..., T(d, k)$. Thus

$$L_f(x)=\sum_{v=1}^{T(d,k)} f(A_v)\, l_{dv}(x). \tag{1.2.3}$$

1.3. The Lebesgue function is defined by

$$\lambda_d(x)=\sum_{v=1}^{T(d,k)} |l_{dv}(x)| \tag{1.3.1}$$

and the Lebesgue constant by

$$\Lambda_d=\sup_{x\in X} \lambda_d(x). \tag{1.3.2}$$

The Lebesgue constant and the Lebesgue function are important invariants of the interpolation process [6, 7]. For example Λ_d is the norm of the operator $f\to L_f$ (where $C(X)$ and π_d^k are both given the sup norm on X).

In the case $X=[-1, 1]\subset \mathbf{R}$ the Lagrange interpolation process has been extensively studied. However, the several variable case is far from being well understood.

In this paper we will consider the case of equally spaced points in the simplex (see Section 2 for the precise definition). We will give precise results for the asymptotic values of the Lebesgue functions (Theorem 4.7) and the Lebesgue constants (Theorem 4.6).

In the one variable case the polynomials l_{dv} (defined in 1.2) have a simple expression as a product of terms. This is not so, in general, in the case of several variables. However, it was observed by L. Bos [2] that for the case of equally spaced points in the simplex, the polynomials l_{dv} have a simple expression as a product and this fact will be exploited in this paper. For this reason, the methods of this paper cannot be expected to work in the general several variable case.

The paper is organized as follows. In Section 2, an expression for $\overline{\lim}(1/d)\,|l_{dv}(x)|$ is given. The starting point for the calculation are the formulae for $l_{dv}(x)$ as a product. In Section 3 we obtain an expression for

$$u(x)=\overline{\lim}\,\frac{1}{d}\log \lambda_d(x)$$

which involves a maximum over a parameter space. We also obtain an expression for $\lim(1/d)\log \Lambda_d$ which involves maximizing $u(x)$ over the simplex. In Section 4 these maxima are explicitly calculated.

The calculations in Sections 2–4 are done in the two variable case, but the generalization to k variables is straightforward and is briefly indicated at the end of Section 4.

Bos [2] obtains an estimate on Λ_d, namely $\Lambda_d \leqslant \binom{2d-1}{d}$. He shows that, as the number of variables increases, $\Lambda_d \to \binom{2d-1}{d}$. In this paper we study Λ_d as $d \to \infty$ with the number of variables fixed.

J. C. Mason [5] has also studied Lebesgue constants and functions for a several variable polynomial interpolation process. The interpolation procedure he studies is not the one defined in 1.1. The procedure he studies is adapted to the case of product sets.

The results of the paper form part of the announcement [1]. In that announcement Theorem 2.7 is incorrect as stated. Lemma 3.2 of this paper is the correct statement.

2. The Simplex in $\mathbf{R}^2$

2.1. Let

$$\Delta = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 \geqslant 0, x_2 \geqslant 0, x_1 + x_2 \leqslant 1\} \tag{2.1}$$

be the unit simplex in $\mathbf{R}^2$.

The equally spaced points of degree d (d an integer $\geqslant 1$) are the points of Δ with coordinates $(n/d, m/d)$ where n and m are integers. Thus $n + m \leqslant d$, $n \geqslant 0$, and $m \geqslant 0$. There are precisely $T(d, 2)$ such points and the considerations of Section 1 apply. For a simplex in general position, the equally spaced points may be defined by barycentric subdivision (see [2]).

Let A_{dv} for $v = 1, ..., T(d, 2)$ denote the equally spaced points of degree d. An explicit formula for the polynomials l_{dv} (see 1.2) has been given by Bos [2]. Namely

$$l_{dv}(x_1, x_2) = \frac{d^d}{n!\, m!\, p!} \prod_{j=0}^{n-1} \left(x_1 - \frac{j}{d}\right) \prod_{j=0}^{m-1} \left(x_2 - \frac{j}{d}\right) \prod_{j=0}^{p-1} \left(x_3 - \frac{j}{d}\right), \tag{2.1.1}$$

where p and x_3 are defined by

$$x_1 + x_2 + x_3 = 1 \qquad \text{and} \qquad n + m + p = d \tag{2.1.2}$$

Also

$$l_{dv}(x_1, x_2) = \sum_{j=0}^{n-1} \left(\frac{x_1 - j/d}{n/d - j/d}\right) \prod_{j=0}^{m-1} \left(\frac{x_2 - j/d}{m/d - j/d}\right) \prod_{j=0}^{p-1} \left(\frac{x_3 - j/d}{p/d - j/d}\right) \tag{2.1.3}$$

so that

$$\log |l_{dv}(x_1, x_2)| = \sum_{j=0}^{n-1} \left(\log \left| x_1 - \frac{j}{d} \right| - \log \left| \frac{n}{d} - \frac{j}{d} \right| \right) + \sum_{j=0}^{m-1} \left(\log \left| x_2 - \frac{j}{d} \right| - \log \left| \frac{m}{d} - \frac{j}{d} \right| \right) + \sum_{j=0}^{p-1} \left(\log \left| x_3 - \frac{j}{d} \right| - \log \left| \frac{p}{d} - \frac{j}{d} \right| \right). \quad (2.1.4)$$

As usual, if n (or m or p) is equal to zero the corresponding portion of the product in (2.1.1) or (2.1.3) is equal to one and the corresponding portion of the sum in (2.1.4) is equal to zero.

2.2. Now consider a sequence of points $A_{d_i v_i}$ (of degree d_i respectively) and suppose they converge to a point $(\alpha, \beta) \in \Delta$. That is $A_{d_i v_i} = (n_i/d_i, m_i/d_i)$ and $\lim_i (n_i/d_i) = \alpha$, $\lim_i (m_i/d_i) = \beta$.

Furthermore we will suppose that

$$\left| \frac{n_i}{d_i} - \alpha \right| = O\left(\frac{1}{d_i}\right) \quad \text{and} \quad \left| \frac{m_i}{d_i} - \beta \right| = O\left(\frac{1}{d_i}\right). \quad (2.2.1)$$

Under these conditions we will calculate $\overline{\lim}_i\, (1/d_i) \log |l_{d_i v_i}(x_1, x_2)|$ (we will suppress the index i).

We thus must estimate sums of the form $(1/d) \sum_{j=0}^{n-1} \log |n/d - j/d|$ and $(1/d) \sum_{j=0}^{n-1} \log |x - j/d|$.

The first sum is essentially a Riemann sum of an integral and the limit is given in Lemma 2.3. The same is true of the second sum in the case of $x > \alpha$ and the limit is given in Lemma 2.5. The second sum, in the case $x < \alpha$ is slightly more complicated and only an upper limit is given (Lemma 2.6).

2.3. Lemma. *Let n_i/d_i converge to $\alpha \in [0, 1]$ under the condition* (2.2.1). *Then*

$$\lim_{d \to \infty} \frac{1}{d} \sum_{j=0}^{n-1} \log \left| \frac{n}{d} - \frac{j}{d} \right| = \int_0^\alpha \log |\alpha - t|\, dt.$$

Proof. The function $\log |n/d - t|$ is negative and decreasing as a function of t on $[0, n/d]$. Thus, comparing the areas of rectangles with the area between a curve and the t-axis we conclude

$$\frac{1}{d} \sum_{j=0}^{n-1} \log \left| \frac{n}{d} - \frac{j}{d} \right| \geqslant \int_0^{n/d} \log \left| \frac{n}{d} - t \right| dt \quad (2.3.1)$$

and

$$\frac{1}{d}\sum_{j=1}^{n-1}\log\left|\frac{n}{d}-\frac{j}{d}\right| \leqslant \int_0^{(n-1)/d}\log\left|\frac{n}{d}-t\right|\,dt. \tag{2.3.2}$$

Thus

$$\frac{1}{d}\sum_{j=0}^{n-1}\log\left|\frac{n}{d}-\frac{j}{d}\right| = \int_0^{n/d}\log\left|\frac{n}{d}-t\right|\,dt + O\left(\frac{\log d}{d}\right). \tag{2.3.3}$$

Lemma 2.3 now follows from the following lemma:

2.4. LEMMA. *Let $0 \leqslant a \leqslant b \leqslant 1$. Suppose that $|a-b| \leqslant c/d$ where c is a constant >0 and d an integer $\geqslant 1$. Then*

$$\int_0^b \log|b-t|\,dt - \int_0^a \log|a-t|\,dt = 0\left(\frac{\log d}{d}\right).$$

Proof. $\int_0^b \log|b-t|\,dt - \int_0^a \log|a-t|\,dt = \int_a^b \log s\,ds$. If $b \leqslant 2c/d$ this is $O(\log d/d)$. If not, $a \geqslant c/d$ and $|\int_a^b \log s\,ds| \leqslant |b-a|\,|\log a|$ which is $O(\log d/d)$.

2.5. LEMMA. *Suppose $\alpha < x \leqslant 1$ and n_i/d_1 converges to $\alpha \in [0,1]$ under condition* (2.2.1). *Then*

$$\lim_{d\to\infty}\frac{1}{d}\sum_{j=0}^{n-1}\log\left|x-\frac{j}{d}\right| = \int_0^\alpha \log|x-t|\,dt.$$

Proof. The proof is similar to Lemma 2.3 and we will not give details.

2.6. LEMMA. *Suppose $\alpha > 0$, $x \in (0,\alpha]$ and n_i/d_i converges to α under conditions* (2.2.1). *Then*

(i) $\overline{\lim}(1/d)\sum_{j=0}^{n-1}\log|x-j/d| = \int_0^\alpha \log|x-t|\,dt$ *and, in fact,*

(ii) $(1/d)\sum_{j=0}^{n-1}\log|x-j/d| \leqslant \int_0^\alpha \log|x-t|\,dt + O(\log d/d)$.

Proof. First note that $\log|x-t|$, as a function of t, is in $L^1[0,\alpha]$ but is not bounded on $[0,\alpha]$.

Let s be the integer such that $|x-j/d|$ is a minimum for $j=0,1,\dots,d$. Suppose $s/d \leqslant x$. (The case $s/d > x$ is handled in an analogous fashion). Then, comparing areas of rectangles with the area between a curve and the t-axis, we have

$$\frac{1}{d}\sum_{j=1}^{s}\log\left|x-\frac{j}{d}\right| \leqslant \int_0^{s/d}\log|x-t|\,dt \tag{2.6.1}$$

and

$$\frac{1}{d}\sum_{j=s+1}^{n-1}\log\left|x-\frac{j}{d}\right|\leqslant\int_{(s+1)/d}^{n/d}\log|x-t|\,dt. \tag{2.6.2}$$

Thus, since $x\leqslant 1$,

$$\frac{1}{d}\sum_{j=0}^{n-1}\log\left|x-\frac{j}{d}\right|\leqslant\int_{0}^{n/d}\log|x-t|\,dt+O\left(\frac{\log d}{d}\right) \tag{2.6.3}$$

and statement (ii) of Lemma 2.6 follows, using Lemma 2.5.

To prove (i) of Lemma 2.6 we must obtain lower bounds for $(1/d)\sum_{j=0}^{n-1}\log|x-j/d|$ for infinitely many values of d. We have

$$\frac{1}{d}\sum_{j=0}^{s-1}\log\left|x-\frac{j}{d}\right|\geqslant\int_{0}^{s/d}\log|x-t|\,dt \tag{2.6.4}$$

and

$$\frac{1}{d}\sum_{j=s+1}^{n-1}\log\left|x-\frac{j}{d}\right|\geqslant\int_{s/d}^{(n-1)/d}\log|x-t|\,dt. \tag{2.6.5}$$

Thus

$$\frac{1}{d}\sum_{\substack{j=0\\ j\neq s}}^{n-1}\log\left|x-\frac{j}{d}\right|\geqslant\int_{0}^{n/d}\log|x-t|\,dt+O\left(\frac{\log d}{d}\right). \tag{2.6.6}$$

Now let $Z_d=\{x\in[0,1]\mid |x-j/d|\geqslant 1/d^3 \text{ for } j=0,1,\ldots,d\}$. If $x\in Z_d$ then $\log|x-s/d|\geqslant -3\log d$, and using (2.6.6) we have

$$\frac{1}{d}\sum_{j=0}^{n-1}\log\left|x-\frac{j}{d}\right|\geqslant\int_{0}^{n/d}\log|x-t|\,dt+O\left(\frac{\log d}{d}\right) \tag{2.6.7}$$

and using Lemma 2.5 we have, for $x\in Z_d$,

$$\frac{1}{d}\sum_{j=0}^{n-1}\log\left|x-\frac{j}{d}\right|\geqslant\int_{0}^{\alpha}\log|x-t|\,dt+O\left(\frac{\log d}{d}\right). \tag{2.6.8}$$

Conclusion (i) of Lemma (2.6) will now follow from (2.6.8), statement (ii), and Lemma 2.7 below

2.7. Lemma. *For $d\geqslant 2$*

$$([0,1]-Z_d)\cap([0,1]-Z_{d+1}) = \left\{x\in[0,1]\,\middle|\, x<\frac{1}{(d+1)^3}\text{ or } x>1-\frac{1}{(d+1)^3}\right\}.$$

Proof. Let $\xi \in ([0,1]-Z_d)\cap([0,1]-Z_{d+1})$. Then we have, for some integer j_0, $0\leqslant j_0\leqslant d$,

$$\left|\xi-\frac{j_0}{d}\right|<\frac{1}{d^3} \tag{2.7.1}$$

and for some integer j_1, $0\leqslant j_1\leqslant d+1$,

$$\left|\xi-\frac{j_1}{d+1}\right|<\frac{1}{(d+1)^3}. \tag{2.7.2}$$

Equations (2.7.1) and (2.7.2) imply that

$$\left|\frac{j_0(d+1)-j_1 d}{d(d+1)}\right|<\frac{1}{d^3}+\frac{1}{(d+1)^3}. \tag{2.7.3}$$

Now $j_0(d+1)-j_1 d$ is an integer. If it is non-zero (2.7.3) is not satisfied for $d\geqslant 2$. If it is zero, then $j_0=j_1=0$ or $j_0=d$ and $j_1=d+1$ and this proves Lemma 2.7.

Conclusion of proof of Lemma 2.6. Using Lemma 2.7 the only points $x\in[0,1]$ not in a set Z_d for infinitely many values of d are $x=0$ or $x=1$. The case $x=0$ is excluded by the hypothesis of Lemma 2.6. In case $x=1$ (and hence $\alpha=1$), (i) of Lemma 2.6 is a special case of Lemma 2.3 (in fact with lim rather than $\overline{\lim}$ in the statement).

2.8. We now introduce the function $H(x,\alpha)$ for $x\in[0,1]$, $\alpha\in[0,1]$ defined by

$$\begin{aligned} H(x,\alpha) &= \int_0^\alpha \log|x-t|\,dt-\int_0^\alpha \log t\,dt \\ &= \begin{cases} x\log x-(x-\alpha)\log(x-\alpha)-\alpha\log\alpha, & \text{for } x\geqslant\alpha \\ x\log x+(\alpha-x)\log(\alpha-x)-\alpha\log\alpha, & \text{for } x\leqslant\alpha. \end{cases} \end{aligned} \tag{2.8.1}$$

Note that $H(x,\alpha)$ is continuous on $[0,1]\times[0,1]$ and is differentiable on $(0,1)\times(0,1)$ for $x\neq\alpha$.

2.9. THEOREM. *Let $A_{d_i v_i}=(n_i/d_i, m_i/d_i)$ be a sequence of points (of degrees d_i) in Δ converging to $(\alpha,\beta)\in\Delta$. Suppose, furthermore that $|n_i/d_i-\alpha|=O(1/d_i)$ and $|m_i/d_i-\beta|=O(1/d_i)$. Let γ and x_3 be defined by*

$$\alpha+\beta+\gamma=1 \qquad \textit{and} \qquad x_1+x_2+x_3=1.$$

Then, if $x_i\neq 0$, for $i=1,2,3$

$$\overline{\lim}\,\frac{1}{d}\log|l_{dv}(x_1,x_2)|=H(x_1,\alpha)+H(x_2,\beta)+H(x_3,\gamma).$$

Proof. Let p_i be defined by $m_i + n_i + p_i = d_i$. Then $|p_i/d_i - \gamma| = O(1/d_i)$. Applying (2.3), (2.5), (2.6), the theorem follows.

2.10. *Remark.* (i) If one of $x_1, x_2, x_3 = 0$ (say $x_1 = 0$) we consider a sequence of the form

$$A_{d_i v_i} = \left(\frac{0}{d_1}, \frac{m_i}{d_i}, \frac{p_i}{d_i}\right), \qquad \text{where } \left|\frac{m_i}{d_i} - \beta\right| = O\left(\frac{1}{d_i}\right)$$

and $p_i + m_i = d_i$.

Then $\overline{\lim}(1/d) \log |l_{dv}| = H(x_2, \beta) + H(x_3, \gamma)$ since in the formula for l_{dv} the product involving x_1 does not occur.

(ii) If $x_1 = 0$ and $A_{d_i v_i}$ converges to $(\alpha, \beta) \in \Delta$ with $\alpha > 0$ then $l_{dv} \equiv 0$ for d sufficiently large since, in this case, the expression for $l_{dv}(0, x_2)$ given in (2.1.1) involves zero factors. (Note that $n > 0$ for d sufficiently large since $\lim(n_i/d_i) > 0$.)

3. Limiting Values for the Lebesgue Functions and Lebesgue Constants

3.1. We introduce the function

$$u(x_1, x_2) = \operatorname*{Max}_{(\alpha, \beta, \gamma) \in W} \{H(x_1, \alpha) + H(x_2, \beta) + H(x_3, \gamma)\}, \tag{3.1.1}$$

where

$$W = \{(\alpha, \beta, \gamma) \mid \alpha \geq 0, \beta \geq 0, \gamma \geq 0, \alpha + \beta + \gamma = 1\} \tag{3.1.2}$$

and $x_1 + x_2 + x_3 = 1$

Note that since H is continuous then u is continuous. In Section 4 we will give an explicit expression for u. In this section we will show how the limiting behaviour of the Lebesgue functions and Lebesgue constants can be given in terms of $u(x_1, x_2)$. The reasoning used in this section is similar to that used by Siciak [8].

Recall that the Lebesgue function is given by

$$\lambda_d(x_1, x_2) = \sum_{v=1}^{T(d,2)} |l_{dv}(x_1, x_2)|.$$

We will denote by $\operatorname{Int}(\Delta)$ the interior of the set Δ introduced in 2.1. That is

$$\operatorname{Int}(\Delta) = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}. \tag{3.1.3}$$

Note that if $(x_1, x_2) \in \mathrm{Int}(\Delta)$ then $x_3 > 0$. Also we will denote by $\mathrm{Bd}(\Delta)$ the boundary of Δ. That is

$$\mathrm{Bd}(\Delta) = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1 = 0,\ x_2 = 0, \text{ or } x_1 + x_2 = 1\}.$$

3.2. Lemma. *For all* $(x_1, x_2) \in \mathrm{Int}(\Delta)$ *then*

$$\overline{\lim} \frac{1}{d} \log \lambda_d(x_1, x_2) = u(x_1, x_2).$$

Proof. Fix $(x_1, x_2) \in \mathrm{Int}(\Delta)$. Suppose that $(\alpha_0, \beta_0, \gamma_0)$ is a point at which the maximum in (3.1.1) is attained. That is

$$u(x_1, x_2) = H(x_1, \alpha_0) + H(x_2, \beta_0) + H(x_3, \gamma_0). \tag{3.2.1}$$

Let $A_{d_i v_i} = (n_i/d_i,\ m_i/d_i)$ be a sequence of points (of degrees d_i) in Δ such that

$$\overline{\lim} \frac{1}{d} |\log l_{dv}(x_1, x_2)| = H(x_1, \alpha_0) + H(x_2, \beta_0) + H(x_3, \gamma_0).$$

By Theorem 2.9 such a sequence always exists for $(x_1, x_2) \in \mathrm{Int}(\Delta)$. Thus

$$\overline{\lim} \frac{1}{d} \log \lambda_d(x_1, x_2) \geqslant u(x_1, x_2). \tag{3.2.2}$$

To prove the opposite inequality we note that it follows from the definition of $\lambda_d(x_1, x_2)$ that

$$\lambda_d(x_1, x_2) \leqslant T(d, 2) \operatorname*{Max}_{v} |l_{dv}(x_1, x_2)| \tag{3.2.3}$$

so that, using (2.6.3) and (2.3.3), we have

$$\frac{1}{d} \log \lambda_d(x_1, x_2) \leqslant \frac{1}{d} \log T(d, 2) + O\left(\frac{\log d}{d}\right) + \operatorname*{Max}_{\substack{m,n,p \\ m+n+p=d}} \left\{ H\left(x_1, \frac{n}{d}\right) + H\left(x_2, \frac{m}{d}\right) + H\left(x_3, \frac{p}{d}\right) \right\}. \tag{3.2.4}$$

Since $T(d, 2) = O(d^2)$ we conclude that

$$\overline{\lim} \frac{1}{d} \log \lambda_d(x_1, x_2) \leqslant \operatorname*{Max}_{(\alpha, \beta, \gamma) \in W} \{H(x_1, \alpha) + H(x_2, \beta) + H(x_3, \gamma)\} = u(x_1, x_2) \tag{3.2.5}$$

and the proof of Lemma 3.2 is complete.

3.3. Lemma. *Let*

$$v(x_2, x_3) = \underset{\substack{\beta+\gamma=1 \\ \beta \geq 0, \gamma \geq 0}}{\operatorname{Max}} \{H(x_2, \beta) + H(x_3, \gamma)\}. \tag{3.3.1}$$

Then, if $x_2 \neq 0$ *and* $x_3 \neq 0$

$$\overline{\lim} \frac{1}{d} \log \lambda_d(0, x_2) = v(x_2, x_3).$$

Proof. The proof uses Remark 2.10 and the methods of 3.2. We omit the details. The analogous results for $\overline{\lim}(1/d) \log \lambda_d$ in case $x_2 = 0$ or $x_3 = 0$ are valid.

Now, after a simple change of variables, one sees that for $x_2 + x_3 = 1$ and $\beta + \gamma = 1$

$$H(x_2, \beta) + H(x_3, \gamma) = \int_0^1 \log |x_2 - t| \, dt - \int_0^\alpha \log t \, dt - \int_0^\beta \log t \, dt. \tag{3.3.2}$$

One deduces that for $x_2 \neq 0$

$$v(x_2, x_3) = F(x_2),$$

where the function F is defined in (4.3.4). The maximum of $F(x_2)$ on $[0, 1]$ is log 2 and we may conclude that

$$\lim \frac{1}{d} \log \Lambda_d \geq \log 2 \tag{3.3.3}$$

if the limit exists.

Essentially then, (3.3.3) is deduced by restricting to the one variable problem.

3.4. Corollary. *If* $(x_1, x_2) \in \operatorname{Bd}(\Delta)$ *then*

$$\overline{\lim} \frac{1}{d} \log \lambda_d(x_1, x_2) \leq u(x_1, x_2).$$

Proof.

$$u(0, x_2) = \underset{\substack{\beta+\gamma \leq 1 \\ \beta \geq 0, \gamma \geq 0}}{\operatorname{Max}} \{H(x_2, \beta) + H(x_3, \gamma)\}$$

and hence $u(0, x_2) \geq v(x_2, x_3)$.

The analogous statements for the other portions of $\operatorname{Bd}(\Delta)$ are valid.

3.5. LEMMA. *Let* $M = \mathrm{Max}_{(x_1, x_2)\in \Delta} u(x_1, x_2)$.

Then $\mathit{lim}(1/d) \log \Lambda_d = M$.

Proof. Recall that $\Lambda_d = \mathrm{Max}_{(x_1, x_2)\in \Delta} \lambda_d(x_1, x_2)$. For each integer $d \geqslant 1$ let (x_1^d, x_2^d) be a point where the maximum of λ_d over Δ is attained. Now, from (3.2.4) and (3.3), (3.4) we have

$$\frac{1}{d}\log \lambda_d(x_1^d, x_2^d) \leqslant u(x_1^d, x_2^d) + O\left(\frac{\log d}{d}\right). \tag{3.5.1}$$

Hence

$$\overline{\lim}\, \frac{1}{d}\log \Lambda_d \leqslant M. \tag{3.5.2}$$

To complete the proof of Lemma 3.5 we will show that $\underline{\lim}(1/d) \log \Lambda_d \geqslant M - \varepsilon$ for any $\varepsilon > 0$.

We proceed as follows. Let (x_1^0, x_2^0) be a point of Δ where u assumes its maximum. That is $u(x_1^0, x_2^0) = M$. Given any $\varepsilon > 0$ and for d sufficiently large there exists (x_1', x_2') close to (x_1^0, x_2^0) such that

$$u(x_1', x_2') \geqslant M - \varepsilon \tag{3.5.3}$$

and x', x_2', x_3' each belong to Z_d.

Let $(\alpha', \beta', \gamma')$ be a point of W such that

$$u(x_1', x_2') = H(x_1', \alpha') + H(x_2', \beta') + H(x_3', \gamma').$$

Using (2.6.8) and the proof of Lemma 2.3 we have for appropriate points A_{dv} that

$$\frac{1}{d}\,|\log l_{dv}(x_1', x_2')| \geqslant H(x_1', \alpha') + H(x_2', \beta') + H(x_3', \gamma') + O\left(\frac{\log d}{d}\right).$$

Since $\Lambda_d \geqslant |l_{dv}(x_1, x_2)|$ for all points $(x_1, x_2) \in \Delta$ we may conclude that

$$\underline{\lim}\, \frac{1}{d}\log \Lambda_d \geqslant M - \varepsilon$$

and the lemma follows.

4. EXPLICIT EXPRESSION FOR $u(x_1, x_2)$

4.1. Formula (3.1.1) gives an expression for $u(x_1, x_2)$ as a maximum over a parameter space. To find an explicit expression for u we must study the following maximum problem.

For x_1, x_2, x_3 fixed, and $(x_1, x_2) \in \operatorname{Int}(\Delta)$, let

$$g(\alpha, \beta, \gamma) = H(x_1, \alpha) + H(x_2, \beta) + H(x_3, \gamma) \tag{4.1.1}$$

We want to find the maximum of g on the set

$$W = \{(\alpha, \beta, \gamma) \mid \alpha \geqslant 0, \beta \geqslant 0, \gamma \geqslant 0, \alpha + \beta + \gamma = 1\}; \tag{4.1.2}$$

g is continuous on W but not differentiable on W. To find the maximum of g on W we will use the standard techniques of differential calculus. First we define certain open subsets of W on which g is differentiable. Namely we define

$$W_3^+ = \{(\alpha, \beta, \gamma) \in W \mid 0 < \alpha < x_1, 0 < \beta < x_2, x_3 < \gamma < 1\} \tag{4.1.3}$$

$$W_3^- = \{(\alpha, \beta, \gamma) \in W \mid x_1 < \alpha < 1,\ x_2 < \beta < 1,\ 0 < \gamma < x_3\} \tag{4.1.4}$$

$$W_2^+ = \{(\alpha, \beta, \gamma) \in W \mid 0 < \alpha < x_1, x_2 < \beta < 1,\ 0 < \gamma < x_3\}, \tag{4.1.5}$$

where W_2^-, W_1^+, W_1^- are defined analogously. If $x_i = 0$ or 1 for $i = 1, 2, 3$ some of the above sets will be empty.

First we look for critical points of g on each set W_i^+ and W_i^- $(i = 1, 2, 3)$ and find the values of g at those critical points. Then we check the values of g on the boundaries of these open sets. Since $W = \bigcup_{i=1}^3 \bar{W}_i^+ \cup \bar{W}_i^-$ the maximum of g on W will be found among the values of g at the critical points in each W_i^+, W_i^- and the value of g on the boundaries of those sets.

4.2. Using Lagrange multipliers, the critical points of g on W_3^+ satisfy

$$\frac{\partial H(x_1, \alpha)}{\partial \alpha} = \frac{\partial H(x_2, \beta)}{\partial \beta} = \frac{\partial H(x_3, \gamma)}{\partial \gamma}. \tag{4.2.1}$$

Using (2.8.1) this becomes

$$\log(x_1 - \alpha) - \log \alpha = \log(x_2 - \beta) - \log \beta = \log(\gamma - x_3) - \log \gamma \tag{4.2.2}$$

or

$$\frac{x_1 - \alpha}{\alpha} = \frac{x_2 - \beta}{\beta} = \frac{\gamma - x_3}{\gamma}. \tag{4.2.3}$$

We let λ denote the common value of the expression in (4.2.3). Note that $\lambda \neq 0$ and we have

$$\begin{aligned} x_1 &= \alpha + \alpha\lambda \\ x_2 &= \beta + \beta\lambda \\ x_3 &= \gamma - \gamma\lambda. \end{aligned} \tag{4.2.4}$$

However, adding, we have

$$1 = x_1 + x_2 + x_3 = (\alpha + \beta + \gamma) + (\alpha + \beta - \gamma)\,\lambda.$$

Since $\lambda \neq 0$ and $\alpha + \beta + \gamma = 1$ it follows that

$$\alpha + \beta = \gamma \tag{4.2.5}$$

and thus

$$\alpha + \beta = \tfrac{1}{2}, \qquad \gamma = \tfrac{1}{2}. \tag{4.2.6}$$

4.3. Equations (4.2.4) and (4.2.6) imply that $x_3 < \frac{1}{2}$ and $x_1 + x_2 > \frac{1}{2}$. If these conditions are not satisfied, there will be no critical point in W_3^+. Assuming there is a critical point, on substituting (4.2.4) into the expression for g we have that the value of g at the critical point is

$$(\alpha + \beta)\{(1 + \lambda)\log(1 + \lambda) - \lambda \log \lambda\} + \gamma\{(1 - \lambda)\log(1 - \lambda) + \lambda \log \lambda\}. \tag{4.3.1}$$

Using (4.2.6) this is equal to

$$\tfrac{1}{2}\{(1 + \lambda)\log(1 + \lambda) + (1 - \lambda)\log(1 - \lambda)\}. \tag{4.3.2}$$

From (4.2.4) we have $\lambda = 1 - 2x_3$ and substituting this into (4.3.2) we have that the value of g at the critical point is

$$(1 - x_3)\log(1 - x_3) + x_3 \log x_3 + \log 2. \tag{4.3.3}$$

We introduce the function

$$F(t) = t \log t + (1 - t)\log(1 - t) + \log 2. \tag{4.3.4}$$

The value of g at the critical point in W_3^+ is $F(x_3)$. An analogous calculation shows that the value of g at the critical point of W_i^+ or W_i^- is $F(x_i)$ for $i = 1, 2, 3$.

We may thus conclude that

$$\operatorname*{Max}_{(\alpha, \beta, \gamma) \in W} g(\alpha, \beta, \gamma) \geqslant \operatorname{Max}(F(x_1), F(x_2), F(x_3)). \tag{4.3.5}$$

4.4. *Remark.* For $0 \leqslant t \leqslant 1$ it is is a simple exercise to see that

$$0 \leqslant F(t) \leqslant \log 2.$$

The maximum occurs when $t = 0$ or 1, the minimum when $t = \frac{1}{2}$.

4.5. THEOREM. *For* $(x_1, x_2) \in \mathrm{Int}(\Delta)$ *then* $u(x_1, x_2) = \mathrm{Max}(F(x_1), F(x_2), F(x_3))$.

Proof. We will prove that one has equality in (4.3.5) by showing that the values of g on the boundaries of W_i^+, W_i^- $(i = 1, 2, 3)$ are bounded by the right hand side of (4.3.5).

We will examine the values of g on the boundary of W_3^+. We will have to consider various cases.

We note that $H(x, \alpha) \equiv 0$ for $\alpha = 0$ or $\alpha = x$ and $H(x, \alpha) \leqslant 0$ for $\alpha \geqslant x$. We first examine that portion of the boundary of W_3^+ where $\alpha = x_1$. It will be convenient to introduce the function

$$\Psi(x_2, x_3, \beta, \gamma) = H(x_2, \beta) + H(x_3, \gamma). \tag{4.4.1}$$

We must maximize this (for x_2, x_3 fixed) subject to $\beta \geqslant 0$, $\gamma \geqslant 0$, $\beta = \gamma = x_2 + x_3$.

Assume $0 < \beta < x_2$ and $x_3 < \gamma < 1$. Using Lagrange multipliers in a similar fashion to the computations of 4.2 and 4.3 we find the value of ψ at the critical point is

$$(x_2 + x_3) \log 2 + x_2 \log x_2 + x_3 \log x_3 - (x_2 + x_3) \log(x_2 + x_3) \tag{4.4.2}$$

and that

$$x_2 > x_3. \tag{4.4.3}$$

Using one variable calculus one finds that the maximum of (4.4.2) considered as a function of x_2 for $x_3 \leqslant x_2 \leqslant 1 - x_3$ is $F(x_3)$, and this maximum occurs at $x_2 = 1 - x_3$. If $\beta = x_2$ then $\gamma = x_3$ and $\psi = 0$. If $\beta = 0$ then $\gamma = x_2 + x_3$ and $\psi = H(x_2, x_2 + x_3) \leqslant 0$.

Next we examine that portion of the boundary where $\alpha = 0$.

Assume $0 < \beta < x_2$. Now

$$\frac{\partial \psi}{\partial x_2} = \log x_2 - \log(x_2 - \beta)$$

and thus

$$\frac{\partial \psi}{\partial x_2} > 0, \qquad \text{if} \quad \beta > 0;$$

ψ is therefore an increasing function of x_2 and, since $x_2 \leqslant 1 - x_3$, we have

$$\psi(x_2, x_3, \beta, \gamma) \leqslant \psi(1 - x_3, x_3, \beta, \gamma). \tag{4.4.4}$$

Also if $\beta=0$ then $\gamma=1$ and $\psi=H(x_3, 1)\leqslant 0$. If $\beta=x_2$ then $\gamma=1-x_2\geqslant x_3$ and $H(x_3, 1-x_2)\leqslant 0$. Thus using the calculations in the previous case

$$\operatorname*{Max}_{\substack{\beta+\gamma=1\\ \beta\geqslant 0,\, \gamma\geqslant 0}} \psi(x_2, x_3, \beta, \gamma)=F(x_3). \tag{4.4.5}$$

This completes the analysis of that portion of the boundary of W_3^+ where $\alpha=0$.

The portion of the boundary of W_3^+ where $\beta=0$ or $\beta=x_2$ is handled analogously. Thus we see that

$$\operatorname*{Max}_{(\alpha, \beta, \gamma)\in \overline{W}_3^+} g(\alpha, \beta, \gamma)=F(x_3).$$

The other sets W_i^+, W_i^- are handled in a similar manner. For example, one may show that g is bounded by $\mathrm{Max}(F(x_1), F(x_2))$ on the boundary of W_3^-.

This completes the proof of Theorem 4.5.

4.6. THEOREM. $\lim_{d\to\infty}(1/d)\log \Lambda_d=\log 2$.

Proof. This follows from Theorem 4.5 and Remark 4.4 and Lemma 3.5.

4.7. THEOREM. *For* $(x_1, x_2)\in \mathrm{Int}(\Delta)$, *then*

$$\overline{\lim}\,\frac{1}{d}\log\lambda_d(x_1, x_2)=\mathrm{Max}(F(x_1), F(x_2), F(x_3)).$$

Proof. This follows from Theorem 4.5 and Lemma 2.3.

4.8. *Remark.* Theorems 4.6 and 4.7 are valid in the case of k-variables. Specifically, let $\Delta=\{x=(x_1,..., x_k)\in \mathbf{R}^k\,|\, x_i\geqslant 0\ i=1, ..., k \text{ and } \sum_{i=0}^k x_i\leqslant 1\}$. The points of Δ of degree d are points with coordinates $(n_1/d, ..., n_k/d)$ where n_i are integers $\geqslant 0$ and $\sum_{i=1}^k n_i=d$. There are $T(d, k)$ such points and the considerations of Section 1 apply. We have

4.7(i). THEOREM. *For* $x\in \mathrm{Int}(\Delta)$ *then* $\overline{\lim}(1/d)\log\lambda_d(x)=\mathrm{Max}(F(x_1), ..., F(x_k))$.

4.6(i). THEOREM. $\lim_{d\to\infty}(1/d)\log \Lambda_d=\log 2$.

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